Math 255B Lecture 22 Notes

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1 Multiplicativity and the Functional Calculus for the Laplacian

1.1 Multiplicativity of the functional calculus

Last time, we were proving the multiplicativity of our functional calculus:

Proposition 1.1. *Let A be self-adjoint, and let $\varphi, \psi \in C_0(\mathbb{R})$. Then $\varphi(A)\psi(A) = (\varphi\psi)(A)$.*

Proof. Last time, we showed that $\operatorname{Im}\varphi(A) \subseteq D(A^j)$ for all j and that for any polynomial p, $p(A)\varphi(A) = (p\varphi)(A)$.

Let $\chi \in C_0(\mathbb{R})$ with $0 \leq \chi \leq 1$ and $\chi = 1$ on $\operatorname{supp}(\varphi) \cup \operatorname{supp}(\psi)$. For $u, v \in H$, write

$$\langle \varphi(A)u, (p\chi)(A)v \rangle = \langle \varphi(A)u, p(A)\chi(A)v \rangle$$

Since $\varphi \in D(A^j)$ for all j,

$$\begin{aligned} &= \langle \overline{p}(A)\varphi(A)u, \chi(A)v \rangle \\ &= \langle (\overline{p}\varphi)(A)u\chi(A)v \rangle . \end{aligned}$$

Take a sequence p_j of polynomials such that $\chi\overline{p}_j \to \chi\psi = \psi$ uniformly on $\mathbb{R}$. Recall that we had for all $f \in C_B$, $\|f(A)\|_{\mathcal{L}(H,H)} \leq 2\|f\|_{L^\infty}$. Thus, $(\chi p_j)(A) \xrightarrow{\mathcal{L}(H,H)} \psi(A) = \psi(A)^*$. Also, $\overline{p}_j\varphi = \overline{p}_j\chi\varphi \to \psi\varphi$ uniformly, so $\overline{p}_j\varphi(A) \xrightarrow{\mathcal{L}(H,H)} (\psi\varphi)(A)$. We get

$$\langle \psi(A)\varphi(A)u, v \rangle = \langle \varphi(A)u, \psi(A)^*v \rangle = \langle (\psi\varphi)(A)u, \chi(A)v \rangle .$$

Now let $\chi \uparrow 1$ pointwise; we claim that $\chi(A) \to 1$ weakly. So we get $\langle \psi(A)\varphi(A)u, v \rangle = \langle (\psi\varphi)(A)u, v \rangle$ for all u, v.

To prove that $\chi(A) \to 1$ weakly, note that if $\varphi_j \in C_B$, $\varphi \in C_B$, and $\varphi_j \to \varphi$ pointwise boundedly ($\exists C$ such that $|\varphi_j(x)| \leq C$ for all j, x), then

$$\langle \varphi_j(A)u, v \rangle = \int \varphi_j(\lambda)\, d\mu_{u,v} \xrightarrow{j \to \infty} \int \varphi(\lambda)\, d\mu_{u,v}(\lambda) = \langle \varphi(A)u, v \rangle$$

by polarization and dominated convergence. $\square$

1.2 Spectrum and functional calculus for the Laplacian

Let $A = -\Delta$ on $L^2(\mathbb{R}^n)$ be self-adjoint with $D(A) = H^2(\mathbb{R}^n)$. Given $\varphi \in C_0(\mathbb{R})$, we compute $\varphi(A)$.

Proposition 1.2. $\operatorname{Spec}(A) = [0, \infty)$.

Proof. First, $\operatorname{Spec}(A) \subseteq [0, \infty)$, as $A \geq 0$: for $u \in D(A)$,

$$\langle Au, u \rangle = \int |\nabla u|^2 \geq 0.$$

To get equality, it suffices to show that $(0, \infty) \subseteq \operatorname{Spec}(A)$ (as the spectrum is closed). For contradiction, let $\lambda > 0$ be such that $A - \lambda : D(A) \to L^2$ is bijective. Then there exists a constant $C > 0$ such that $\|u\|_{L^2} \leq C \|(A - \lambda)u\|_{L^2}$ for any $u \in D(A)$.

Remark 1.1. A has no eigenvalues:[1] If $u \in L^2$ and $(-\Delta - \lambda)u = 0$, then taking the Fourier transform, we get

$$(|\xi|^2 - \lambda)\widehat{u}(\xi) = 0,$$

so $\widehat{u} = 0 \implies u = 0$.

Instead, we want to find **generalized eigenfunctions** $u \in L^\infty$ such that $(-\Delta - \lambda)u = 0$. We can take $u(x) = e^{ix \cdot \xi}$ for $\xi \in \mathbb{R}^n$ (where $|\xi|^2 = \lambda$). Consider the **quasimodes**[2] $u_j(x) = j^{-n/2}\chi(x/j)e^{ix\xi}$, where $\chi \in C_0^\infty(\mathbb{R}^n)$ is 1 near 0 with $\|\chi\|_{L^2} = 1$. Then $\|u_j\|_{L^2} = 1$, and

$$\|(A - \lambda)(j^{-n/2}\chi(x/j)e^{ix\cdot\xi})\|_{L^2} = O(1/j).$$

So the lower bound inequality for $A - \lambda$ cannot hold.

To determine $\varphi(A)$ for $\varphi \in C_0(\mathbb{R})$, notice that $\varphi(A) = 0$ if $\operatorname{supp}(\varphi) \subseteq (-\infty, 0)$. Compute the resolvent first: If $\operatorname{Im} z \neq 0$,

$$R(z)u = v, \quad u, v \in L^2 \iff (-\Delta - z)v = u$$

By Fourier transform, we get

$$R(z)u = \mathcal{F}^{-1}\left(\frac{\widehat{u}(\xi)}{|\xi|^2 - z}\right).$$

We get

$$\langle \varphi(A)u, u \rangle = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int \varphi(\lambda) \left\langle (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon))u, u \right\rangle \, d\lambda$$

[1] This says that the spectral measures have no pure point components.

[2] This terminology is common in mathematical physics literature.

By Parseval,

$$= \lim_{\varepsilon\to 0^+} \frac{1}{(2\pi)^n} \lim_{\varepsilon\to 0^+} \iint \varphi(\lambda)|\widehat{u}(\xi)|^2 \left(\frac{1}{|\xi|^2-\lambda-i\varepsilon} - \frac{1}{|\xi|^2-\lambda+i\varepsilon}\right) d\lambda\, d\xi$$

Integrate first in λ and send $\varepsilon \to 0$ using dominated convergence.

$$= \frac{1}{(2\pi)^n} \int \varphi(|\xi|^2)|\widehat{u}(\xi)|^2\, d\xi.$$

So we get

$$\varphi(A)u = \mathcal{F}^{-1}(\varphi(|\xi|^2)\widehat{u}). \qquad \square$$

1.3 Correcting the norm bound in the functional calculus

Proposition 1.3. *Let A be self-adjoint, and let $\varphi \in C_0(\mathbb{R})$.*

$$\|\varphi(A)\|_{\mathcal{L}(H,H)} \le \|\varphi\|_{L^\infty}.$$

Previously, we had a factor of 2 in the bound.

Proof. We have

$$\begin{aligned} \|\varphi(A)u\|^2 &= \langle \varphi(A)u, \varphi(A)u\rangle \\ &= \langle \overline{\varphi}(A)\varphi(A)u, u\rangle \\ &= \left\langle |\varphi|^2(A)u, u\right\rangle \\ &= \int |\varphi|^2(\lambda)\, d\mu_u(\lambda) \\ &\le \|\varphi\|_{L^\infty}^2 \|u\|^2. \end{aligned} \qquad \square$$

We also get the following result.

Corollary 1.1.

$$\varphi(A)\|_{\mathcal{L}(H,H)} \le \|\varphi\|_{L^\infty(\mathrm{Spec}(A))}.$$

Next, we will extend this multiplicativity property to more continuous functions.